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STRUCTURE OF LORENTZIAN TORI WITH A KILLING VECTOR FIELD

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ABSTRACT. All Lorentzian tori with a non-discrete group of isometries are characterized and explicitly obtained. They can lie into three cases: (a) flat, (b) conformally flat but non-flat, and (c) geodesically incomplete. A detailed study of many of their properties (including results on the logical dependence of the three kinds of causal completeness, on geodesic connectedness and on prescribed curvature) is carried out. The incomplete case is specially analyzed, and several known examples and results in the literature are generalized from a unified point of view.

1. INTRODUCTION

The purpose of this article is to study in detail Lorentzian tori admitting an isometry group of dimension greater than zero, with a double aim. Our first aim is to obtain them explicitly (Theorems 2.3, 4.2, 4.5, Lemma 4.4 and Corollary 4.6; see Section 2 for notation) showing the next result.

Structure Result (SR). *The dimension of the isometry group* $\mathrm{Iso}(T^2, g)$ *of a Lorentzian torus* (T^2, g) *is less than or equal to* 2*, with equality if and only if* g *is flat.*

If (T^2, g) *admits a (non-trivial) Killing vector field* ξ*, then* ξ *does not vanish at any point and:*

(1) The metric g *is flat if and only if* $g(\xi, \xi)$ *is constant.*

(2) The metric g *is conformally flat if and only if* $g(\xi, \xi)$ *has a definite sign (strictly positive, strictly negative or identically zero), and if and only if* g *is geodesically complete (in the three causal senses).*

Moreover, g *is conformally flat but non-flat if and only if it is, up to a (finite) covering, isometric to one of the* $\mathcal{G}^c$*-tori constructed in Section* 3*.*

(3) The metric g *is non-conformally flat if and only if* g *is geodesically incomplete in the three causal senses, and if and only if it is, up to a covering, isometric to one of the* $\mathcal{G}$*-tori constructed in Section* 3*.*

Even more, in the case (3) and the non-flat ones of (2), $\mathrm{Iso}(T^2, g)$ *is compact.*

It is worth pointing out, about **SR**:

(A) Hopf-Rinow's theorem yields the equivalence between geodesic completeness and metric completeness with respect to the distance canonically associated to any

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Riemannian metric g_R; in particular, compact Riemannian manifolds are always (geodesically) complete. But there is neither distance associated to a Lorentzian metric nor any analogous conclusion to those of Hopf-Rinow's theorem. Clifton-Pohl's torus is a well-known example of compact Lorentzian surface (geodesically) incomplete. Even more, geodesics of the three causal types (timelike, spacelike and null) can be considered, and there are (non-compact) examples showing the absolute logical independence among the completeness of geodesics of different causal type (see, for instance [BE, Theorem 5.4]). Nevertheless, there are no examples showing such logical independence in the compact case. In fact, a certain "generic" dependence on a torus is shown in [CR]; but there it is also argued that, probably, the three kinds of completeness should be independent on some pathological Lorentzian tori. From **SR** the absolute logical dependence among the three kinds of causal completeness is obtained for Lorentzian tori with $\dim(\mathrm{Iso}(T^2, g)) > 0$.

(B) For any surface endowed with a Riemannian metric g_R, isothermal coordinates can be found, yielding the locally conformal flatness of g_R [Ka, pp. 34-35]. By Gauss-Bonnet's Theorem, if the surface is compact and orientable then the Riemannian metric can be globally conformally flat just when the surface is, topologically, a torus. And, by standard elliptic theory, all the Riemannian metrics on a torus $T^2 \equiv \mathbb{R}^2/\mathbb{Z}^2$ are globally conformally flat. Moreover, for each smooth function f on a Riemannian torus (T^2, g_R) with $\int_{T^2} f\mu_R = 0$ there exists another smooth function u on T^2 such that:

$$\Delta_R u = f. \tag{1.1}$$

(μ_R, Δ_R denote the volume element and Laplacian associated to g_R, respectively). In particular, if f is equal to the Gaussian curvature of the metric, the globally conformal flatness of g_R is obtained. On the other hand, it is easy to show the existence of isothermal coordinates for Lorentzian surfaces (locally, Lorentzian metrics on surfaces look like $\varphi(x, y)(dx \otimes dy + dy \otimes dx)$, $\varphi > 0$, in suitable coordinates). For the compact ones, Gauss-Bonnet's Theorem still holds (see, for instance, [BN] or [J]) and, as a compact manifold admits a Lorentzian metric if and only if its Euler characteristic is zero, the only compact orientable surfaces admitting Lorentzian metrics must be tori. But not all the Lorentzian tori are globally conformally flat. It can be checked because conformally flat metrics on T^2 are complete, but null-incompleteness is a conformal invariant [RS4]. By **SR**, when $\dim(\mathrm{Iso}(T^2, g)) > 0$, the completeness of the metrics allows us to distinguish between conformally and non-conformally flat Lorentzian tori. For these last tori, the analogous equation to (1.1) for the Gaussian curvature does not admit any solution (see also Section 9 later on).

(C) A compact Riemannian manifold must have a compact isometry group, but it does not hold for a compact Lorentzian manifold, in general. D'Ambra [D] showed that a compact 1-connected analytic Lorentzian manifold must have a compact isometry group. He also pointed out that the analyticity is just a technical assumption, and it would be interesting to know when it is necessary. Nevertheless, a flat Lorentzian torus may have a non-compact isometry group (see [D] or Remark 4.3, later on). Thus, from **SR**, the connected part of the identity of $\mathrm{Iso}(T^2, g)$ is always compact, and $\mathrm{Iso}(T^2, g)$ is compact when it has dimension 1.

To get **SR**, we first characterize in Section 2 conformally flat metrics on T^2 as those having a timelike conformal vector field, Theorem 2.3 (compare this section with [H]). In Section 3 we introduce two families $\mathcal{G}, \mathcal{G}^c$ of metrics on T^2 which will be sufficiently representative of the non-conformally and conformally flat metrics having a Killing vector field. For technical reasons, we focus our attention on a subset $\mathcal{G}'$ of $\mathcal{G}$-metrics, whose properties are analyzed in the Appendix. In particular, all the metrics in $\mathcal{G}'$ are shown to be incomplete. We also consider three subfamilies, $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ of $\mathcal{G}$ each one containing metrics of similar properties. In Section 4, the proof of **SR** is completed. To get it, two preliminary steps are given: (a) to show that a (non-trivial) Killing vector field on a Lorentzian torus can not vanish at any point, Lemma 4.1, and (b) to reduce the problem to the study of a $\mathcal{G}$ or $\mathcal{G}^c$-metric. For this step, we prove in particular that if $\dim(\mathrm{Iso}(T^2, g)) = 1$ the integral curves of a Killing vector field are closed, see Lemma 4.4 and below (note that a general result in [DG] shows the same conclusion for compact, 1-connected and—by technical reasons again—analytic Lorentzian manifolds). In Counterexample 4.7 we show how these results fail for conformal vector fields, and in Remark 4.8 we compare them with the Riemannian ones.

With this machinery, we develop our second aim, which is to study several properties of Lorentzian tori admitting a (non-trivial) Killing vector field. The assumption of admitting a Killing vector field on a compact Lorentzian manifold has been used in the literature from different points of view. It can help to supply the loss of compactness of the isometry group and to make easier the classification of spaces of constant curvature, [Ku1], [K], [RS5]. There exist beautiful examples admitting this vector field [KR] and stationary spacetimes (the name in Physics of those spacetimes admitting a timelike Killing vector field), has been widely studied in General Relativity [SW]. We will pay attention specially to incomplete Lorentzian tori. In particular, all the examples of incomplete Lorentzian tori in the literature, except those in [CR], are $\mathcal{G}$-tori, and new properties of them are explored. We can summarize our study as follows. In Section 5 geodesic connectedness is considered. Recall that, even though complete Riemannian metrics are geodesically connected, it does not hold for Lorentzian ones. In fact, the standard Lorentzian pseudosphere $S_1^n, n \geq 2$, is geodesically complete but not geodesically connected, [O, Chapter 9]. Nevertheless, it is a long standing open question if there exist compact counterexamples of complete but not geodesically connected Lorentzian surfaces, [Sp, pp. 8-55]. Beem gave [B, Section 3] an example of incomplete and geodesically disconnected torus. We will obtain families of incomplete tori which are as geodesically connected as geodesically disconnected from $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ metrics, Theorems 5.1, 5.4. In particular, Beem's example is extended to all $\mathcal{G}_3$-metrics. Moreover, Clifton-Pohl's torus is shown to be in this family of metrics, being essentially isometric to Beem's quoted example, Remark 5.2.

In Section 6 we consider null geodesics. Recall that in any orientable and time-orientable Lorentzian surface, there exist two foliations yielded by null geodesics. Null completeness can be characterized in terms of these foliations when one of them is a foliation by circles, [CR, Théorème 2.1]. In Proposition 6.1 the completeness of the null geodesics of each foliation is directly characterized for $\mathcal{G}$-metrics. In Remark 6.2 the topological properties of the subsets of complete and incomplete geodesics shown in [RS1] are generalized by $\mathcal{G}_1$ and $\mathcal{G}_2$ metrics. The existence of closed geodesics is studied in Section 7, characterizing the existence of closed null geodesics and null foliations by circles, Theorem 7.1. Moreover, Galloway [G] gave an example of torus without timelike closed geodesics, and we give a new subfamily $\mathcal{G}_{gal}$ of $\mathcal{G}$-metrics which generalize Galloway's torus (Remark 7.2(1)). In Section 8 we show the existence of complete geodesics in $\mathcal{G}$-tori with velocity not contained in any compact subset of the tangent bundle, which clearly can not occur in Riemannian tori.

We end Section 9 with several consequences on the solvability of (inhomogeneous) D'Alembert's equation on a torus, Corollary 9.1, Remark 9.2, (that is, the analogous equation to (1.1) in the Lorentzian case, obtained replacing the Laplacian by the D'Alembertian, which has the same formal definition). This equation has been studied in the non-compact case in [BP]. But our results are closer to those of the next related problem.

(a) Kazdan and Warner [KW] studied the equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = -k \exp(2u), \tag{1.2}$$

for each smooth function $k \not\equiv 0$ on T^2. They showed that (1.2) admits a solution if and only if: (i) the sign of k changes (that is, k is strictly positive at some point and strictly negative at another one) and (ii) $\int_{T^2} k\mu_0 < 0$, for the canonical volume element μ_0. As a consequence, only functions satisfying (i) and (ii) are the curvature functions for metrics globally pointwise conformal to the usual on T^2.

(b) Burns [Bu] studied a result analogous to Kazdan and Warner's for the Lorentzian case. First, he considered the solvability on T^2 of

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = -k \exp(2u), \tag{1.3}$$

and he found that the quoted conditions (i) and (ii) for k are not enough to obtain solutions. Nevertheless, if k depends on just the first variable, then the conditions (i) and (ii) are necessary and sufficient. This result determines which functions of a single variable can be Gaussian curvatures of these Lorentzian metrics on T^2 which are globally pointwise conformal to the usual one, $dx_1^2 - dx_2^2$ [Bu, Theorem 2.2]. Then, he obtained that if k satisfies just condition (i) (and it is independent of the second variable) there exists a conformally flat metric with curvature k [Bu, Theorem 2.3].

In Theorem 9.4, under this same assumption, we will construct explicitly flat metrics with prescribed curvature k which are as conformally flat as non-conformally flat.

Finally, it is worth pointing out that, among the metrics admitting a non-vanishing Killing vector field, conformally flat ones coincide with stationary metrics. So, we can restate several results above as:

A Lorentzian torus is conformally flat if and only if it is pointwise conformal to a stationary torus, and a Lorentzian torus admitting a non-vanishing Killing vector field is complete if and only if it is stationary.

Some properties of stationary spacetimes, and especially of their geodesics, have been recently studied in [BF], [GM], [H], [M]. Our study of the non-conformally flat $\mathcal{G}$-metrics can be extended to stationary tori, and, so, affirm: (1) all such tori are geodesically connected, (2) Theorem 7.1 on closed geodesics still holds and, (3) in general, quite a few examples can be obtained for several properties of geodesics in stationary spacetimes. Note also that, even though compact spacetimes do not have a good causal behaviour, the universal coverings of the $\mathcal{G}$- or $\mathcal{G}^c$-tori obtained in most cases (for instance, when E in (3.1) vanishes identically) have the best possible causal behaviour, that is, they are globally hyperbolic. For several reasons justifying the importance in Physics of Lorentzian tori, see, for example, [Y] and references therein.

2. Conformally flat metrics

Let (M, h) be a semi-Riemannian manifold, that is, a manifold M endowed with a non-degenerate metric tensor h (we will assume implicitly C^2-differentiability for metrics). A vector X_p tangent to M at p is timelike if $g(X_p, X_p) < 0$, null if $g(X_p, X_p) = 0$, and spacelike if $g(X_p, X_p) > 0$. Recall that two non-degenerate metrics h, h^* on M are said to be conformal if there exists a diffeomorphism $\phi : M \longrightarrow M$ such that the pull-back by ϕ of h^* satisfies $\phi^* h^* = \mu h$ for some positive function μ on M. If ϕ can be chosen as the identity, the metrics are called pointwise conformal. The metric h is conformal to a flat metric if and only if it is pointwise conformal to a flat metric; in this case we say that h is conformally flat. A conformal vector field on (M, h) is a vector field ξ such that

$$\mathcal{L}_\xi h = \sigma h, \tag{2.1}$$

for some function σ on M ($\mathcal{L}$ denotes the Lie derivative). Note that, clearly, if ξ is a Killing vector field (i.e. $\sigma \equiv 0$) for h then, for any pointwise conformal metric $h^* = \mu h$, we have: $\mathcal{L}_\xi(h^*) = \xi(\log \mu) h$, and thus ξ is conformal for h^*.

Lemma 2.1. *Let ξ be a conformal vector field for the semi-Riemannian manifold (M, h) which is non-null at any point. Then ξ is a (unitary) Killing vector field for the pointwise conformal metric $h^* = (1/|h(\xi, \xi)|)h$.*

Proof. Let σ be the function on M satisfying (2.1) and take $\varepsilon \in \{\pm 1\}$ such that $\varepsilon h(\xi, \xi) > 0$. Then:

$$\begin{aligned} \mathcal{L}_\xi h^* &= -\varepsilon (1/h(\xi,\xi)^2)\xi(h(\xi,\xi))h + \varepsilon(1/h(\xi,\xi))\sigma h \\ &= \big(-(1/h(\xi,\xi)^2)(\mathcal{L}_\xi h)(\xi,\xi) + (1/h(\xi,\xi))\sigma\big)\, \varepsilon h = 0. \end{aligned} \tag{2.2}$$

□

Lemma 2.2. *Let (M, g) be a Lorentzian connected surface admitting a Killing vector field $\xi (\not\equiv 0)$. If $g(\xi, \xi)$ is constant then ξ is parallel and the metric g is flat.*

Proof. Let $p \in M$ be a point such that $\xi_p \neq 0$, and take a non-vanishing vector field ξ' in a neighbourhood U of p such that $g(\xi', \xi')$ is constant and (a) if ξ_p is null, $g(\xi, \xi') = 1$, (b) otherwise $g(\xi, \xi') = 0$. Then, the constancy of $g(\xi, \xi)$ shows that $\nabla_X \xi$ is proportional to ξ in case (a) and to ξ' in case (b), for all vector field X on U. Then, by a straigthforward computation, ξ and ξ' are parallel, and, thus, g is flat on U. By a standard argument, the set of points in which ξ must be parallel is open and closed. □

So, if the non-trivial vector field ξ is null, then g is flat. Now, we can characterize conformally flat Lorentzian tori as follows.

Theorem 2.3. *For a Lorentzian torus (T^2, g) it is equivalent: (i) to be (globally) conformally flat and (ii) to admit a timelike (or spacelike) conformal vector field.*

(Of course, if g is conformally flat it admits a (non-trivial) conformal null vector field, but the converse is not true, see Counterexample 4.7.)

Proof. (i)⇒(ii) Let $\mu > 0$ be such that $g^* = \mu g$ is flat and, then, complete. The universal semi-Riemannian covering of (T^2, g^*) must be the two dimensional Lorentz-Minkowski spacetime $\mathbb{L}^2$. Thus, any deck transformation f of $\mathbb{L}^2$ is composition of translations and vector isometries, that is, $f(x) = Ax + b$, $\forall x \in \mathbb{L}^2$, with $b \in \mathbb{L}^2$ and A a vector isometry of $\mathbb{L}^2$. Then 1 must be an eigenvalue of A, otherwise f would leave a fixed point. As A must also be orientation-preserving, necessarily A is the identity, and the natural coordinate vector fields can be induced in parallel fields for (T^2, g^*).

(ii)⇒(i) If ξ is such a conformal vector field of g, consider the metric $g^* = (1/|g(\xi,\xi)|)g$. By Lemmas 2.1 and 2.2, g^* is flat, as required. □

Observe that, in particular, conformally flat Lorentzian tori are time-orientable; recall that orientability and time-orientability are logically independent (see, for instance, [O, Chapter 5]). Of course, Theorem 2.3 holds if we replace T^2 by $\mathbb{R}^2$. From the next section it will be easy to obtain non-conformally flat tori with a non-vanishing Killing vector field which is nonspacelike at any point, and is timelike almost everywhere.

3. Families of non-flat metrics with a Killing vector field

Consider, in usual coordinates, the metric on $\mathbb{R}^2$

$$g_{(x_1,x_2)} = E(x_1)dx_1^2 + F(x_1)[dx_1 \otimes dx_2 + dx_2 \otimes dx_1] - G(x_1)dx_2^2, \tag{3.1}$$

where $E, F, G \in C^2(\mathbb{R})$ satisfy the conditions: (i) $EG + F^2 > 0$, that is, g is Lorentzian, and (ii) E, F and G are periodic with the same period 1, so, the metric is naturally inducible on a torus T^2 by using a unit translation $T_{(1,0)}$ in the x_1 direction, and a non-zero translation $T_{(0,b)}$ in the x_2 one. (This family was also introduced in [RS2] by several heuristic arguments). As the vector field $\xi = \partial/\partial x_2$ is Killing, if $|G|$ were greater than 0 at every point, then ξ would be timelike or spacelike, and, by Theorem 2.3 this metric would be conformally flat; moreover, if $G \equiv 0$ then g is flat (Lemma 2.2). Thus, denote by $\mathcal{G}^c$ the set of metrics given by (3.1) and satisfying (i), (ii) and also: (iii)^c the function G is not constant and $|G| > 0$; the corresponding tori yielded by the translations $T_{(1,0)}, T_{(0,1)}$ will be called $\mathcal{G}^c$-tori. In the next section we will see that the $\mathcal{G}^c$-tori are all the non-flat conformally flat Lorentzian tori, up to a covering. In the remainder of this section, we are going to focus our attention on the family of metrics $\mathcal{G}$ (and corresponding $\mathcal{G}$-tori) in which G has non-constant sign (strictly positive, strictly negative or identically 0), that is, G will consist of the set of metrics given by (3.1) satisfying (i), (ii) and also: (iii) G has non-constant sign. We will see that condition (iii) implies geodesic null-incompleteness, and so, these metrics are not conformally flat. In the next section we will see that, in fact, all non-conformally flat tori admitting a non-trivial Killing vector field are, up to a covering, $\mathcal{G}$-tori.

If $\gamma : [0, b[\longrightarrow \mathbb{R}^2$, $0 < b \leq \infty$, $\gamma(t) = (x_1(t), x_2(t))$, is a geodesic of $g \in \mathcal{G}$, then $g(\gamma', \gamma') = C$, $g(\gamma', \xi) = D$ for some $C, D \in \mathbb{R}$, and

$$\begin{gathered} x_1'(t) = \varepsilon\sqrt{\frac{D^2 + GC}{EG + F^2}}(x_1(t)), \\ G(x_1(t))x_2'(t) = F(x_1(t))x_1'(t) - D, \end{gathered} \tag{3.2}$$

where $\varepsilon \in \{-1, 1\}$. To fix ideas, the second equation (3.2) suggests considering the subfamily $\mathcal{G}'$ of $\mathcal{G}$ containing the metrics satisfying: (iii') $G(0) = 0$, and G has only isolated zeros $p_0 = 0, p_1, \dots, p_{n-1} \in]0,1[$, $p_n = 1$, $n \geq 1$, $p_{k+n} = p_k + 1$ for all integer k. This property, although technically convenient, is not essential, and the results we will obtain for $\mathcal{G}'$-metrics can be extended for $\mathcal{G}$-ones just by taking into account that a $\mathcal{G}$-metric behaves as a $\mathcal{G}'$-one whenever $|G| > 0$, and with several tedious modifications in the proofs.

On the other hand, observe that even though all the geodesics satisfy equations (3.2), not all the solutions of these equations are geodesics. This is due to the loss of unicity in the solutions of (3.2) (fixed initial conditions) which happens: (a) when x_1' vanishes (the square root in (3.2) is not locally lipschitzian at 0), and (b) when $G \circ x_1$ vanishes. To solve this problem, we must consider the equation of the geodesics, and take into account the next relations (in addition to (3.2)) for $\mathcal{G}'$-metrics:

$$\begin{aligned} x_1''(t_1) + (1/2)\left(GG'/(EG+F^2)\right)(x_1(t_1))x_2'(t_1)^2 = 0, \quad &\text{if} \quad x_1'(t_1) = 0, \\ x_2'' + (G'(p_i)/2F(p_i))(x_2')^2 = 0, \quad &\text{if} \quad G \circ x_1 = 0. \end{aligned} \tag{3.3}$$

A direct computation from equations (3.2) and (3.3) yields all the properties (completeness, asymptotical or qualitative behavior) of $\mathcal{G}'$-geodesics. These properties are summarized in the Appendix. In particular, we can state that any geodesic γ of a Lorentzian metric g in $\mathcal{G}'$, with $D^2 + GC > 0$ and $\varepsilon D/F(0) < 0$, is incomplete, and we have, by a straightforward extension of the result to all $\mathcal{G}$-metrics:

Proposition 3.1. $(\mathbb{R}^2, g)$ *is spacelike, null and timelike incomplete, for all g in $\mathcal{G}$.*

Finally, for posterior reference, consider also among the metrics in $\mathcal{G}'$ the next three subfamilies:

$$\mathcal{G}_1 = \{g \in \mathcal{G}'/G|_{]0,1[} > 0\}, \tag{3.4a}$$

$$\mathcal{G}_2 = \{g \in \mathcal{G}'/G'(p_i) \neq 0, F(p_i)F(0) > 0, 0 \leq i \leq n-1\}, \tag{3.4b}$$

$$\mathcal{G}_3 = \{g \in \mathcal{G}'/G'(p_i) \neq 0, F(p_i)F(p_{i+1}) < 0, 0 \leq i \leq n-1\}. \tag{3.4c}$$

4. Characterization of Lorentzian tori by their isometry group

Lemma 4.1. *Let $\xi(\not\equiv 0)$ be a Killing vector field on a connected Lorentzian surface (M, g). Then the set of zeroes of ξ is discrete and the Hopf index of ξ at each zero is -1. Thus, if M is compact then ξ does not vanish at any point.*

Proof. Let p be a zero of ξ, and take a coordinate neighborhood $(U, (x_1, x_2))$ centered at p such that, $g = \lambda(x_1, x_2)[dx_1 \otimes dx_2 + dx_2 \otimes dx_1]$, and $\partial\lambda/\partial x_i = 0$ at 0. Then, ξ can be regarded as an application $\xi : U' \longrightarrow \mathbb{R}^2$, with U' a neighborhood of 0 in $\mathbb{R}^2$. Let ∇ be the Levi-Civita connection of g, and consider a local one-parameter group $\{\phi_t\}$ of ξ at p. For each tangent vector at p, $X \in T_pM$, we can write $d\phi_t X = X + tAX + O(t^2)$, where $O(t^2)$ represents second-order terms in t, and A denotes the linear map:

$$AX = \lim_{t\to 0} \frac{d\phi_t(X) - X}{t} = [X, \xi] = \nabla_X \xi \equiv X(\xi),$$

(the last equality by our choice of coordinates). Deriving the relation $(\phi_t^*(g))(X, Y) = g(X, Y)$, with respect to t we have: $g(X, AY) + g(AX, Y) = 0$, $\forall X, Y \in T_pM$. Therefore, the matrix of A in our coordinates must be

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

for some $a \in \mathbb{R}$, which coincides with the Jacobian of ξ at 0. Thus, if $a \neq 0$, p is an isolated zero of ξ with Hopf index -1. Clearly, the case $a = 0$ can not hold, because we would have $\xi_p = 0, (\nabla\xi)_p = 0$, and, as ξ is Killing, it would follow $\xi \equiv 0$.

For the last assertion, note that if M is also assumed to be compact then its Euler characteristic must be zero; thus, use the classical Poincaré-Hopf theorem. □

Lemma 4.1 allows us to easily prove the next characterization of flat tori.

Theorem 4.2. *(1) The isometry group of a Lorentzian torus* $\mathrm{Iso}(T^2, g)$ *has dimension smaller than or equal to 2, and equality holds if and only if g is flat.*

(2) Let (T^2, g) *be a Lorentzian torus admitting a Killing vector field* $\xi(\not\equiv 0)$. *Then g is flat if and only if* $g(\xi, \xi)$ *is constant.*

Proof. (1) Observe first that if there exist several (globally) independent Killing vector fields ξ_i on T^2 then they must also be pointwise independent; otherwise, if they are linearly dependent at $p \in T^2$ we could construct a new Killing vector field $\xi \not\equiv 0$, which is a linear combination of the ξ_i and vanishing at p, in contradiction with Lemma 4.1. In particular, $\dim(\mathrm{Iso}(T^2, g)) \leq 2$.

If equality holds then any two independent Killing vector fields ξ, ξ' must be pointwise independent. Now, consider the subset $N \subset T^2$ containing the points with zero curvature. Clearly, N is closed and, by the Gauss-Bonnet Theorem, non-empty. Choose any $q \in N$ and take the one-parameter groups of isometries ϕ, ϕ' associated to ξ, ξ', and construct the mapping $\mathbb{R} \times \mathbb{R} \longrightarrow T^2$, $(s, t) \longrightarrow \phi_s \circ \phi'_t(q)$. The differential of this mapping at the origin can be naturally identified with $\{\xi_p, \xi'_p\}$, thus, it is non-singular. As ϕ_s, ϕ'_t preserve the curvature, a neighborhood of q lies in N, and N must be open. This proves the necessary condition and the converse is trivial.

(2) If g is flat its universal Lorentzian covering is $\mathbb{L}^2$, which admits two independent parallel vector fields which may be induced on T^2 (see the proof of Theorem 2.3). As $\dim(\mathrm{Iso}(T^2, g)) = 2$, every Killing vector field on T^2 must be a linear combination of this two parallel vector fields. For the converse, use Lemma 2.2. □

Remark 4.3. Clearly, we have also proved that, for a flat torus, the connected part of the identity in $\mathrm{Iso}(T^2, g)$ is isomorphic to $S^1 \times S^1$. Nevertheless, $\mathrm{Iso}(T^2, g)$ may have infinitely many connected parts and, so, may be non-compact. As an example, consider on $\mathbb{R}^2$ the flat Lorentzian metric $g = dx_1^2 + (dx_1 \otimes dx_2 + dx_2 \otimes dx_1) - dx_2^2$, inducible on T^2. The linear map B, with naturally associated matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix},$$

is an isometry for g also inducible on T^2 , and the closed subgroup $\{B^n : n \in \mathbb{Z}\} \subset \mathrm{Iso}(T^2, g)$ is not compact.

Now, we are going to see what happen if $g(\xi, \xi)$ is not constant. Observe that by classical Poincaré-Bendixon-Schwartz Theorem, if ρ is an inextendible integral curve of ξ then either its range is dense in T^2 either in the adherence of its range there exists a closed integral curve of ξ. As $g(\xi, \xi)$ is not constant such closed integral curves must exist. Next, we are going to see that, in fact, all the integral curves of ξ must be closed; so, we will be able to see g as a $\mathcal{G}$- or $\mathcal{G}^c$-metric.

Lemma 4.4. *Let* (M, g) *be a compact Lorentzian manifold such that* $\dim(\mathrm{Iso}(M, g))$ $= 1$ *and one of its Killing vector fields* ξ *satisfies* $g(\xi, \xi)_p < 0$ *for some* $p \in M$. *Then* $\mathrm{Iso}(M, g)$ *is compact.*

Proof. Consider the subset $\mathcal{C}$ of the tangent bundle TM defined by

$$\mathcal{C} = \{\xi_q/(-g(\xi_q, \xi_q))^{1/2} \in TM/g(\xi_q, \xi_q) = g(\xi_p, \xi_p)\},$$

and the corresponding orthogonal sphere bundle

$$\mathcal{C}' = \{v_q \in TM/\xi_q \in \mathcal{C}, g(v_q, \xi_q) = 0, g(v_q, v_q) = 1\}.$$

Clearly $\mathcal{C} \cup \mathcal{C}'$ is a compact subset of TM, and all the orthonormal basis lying in $\mathcal{C} \cup \mathcal{C}'$ yields a compact subset $\mathcal{D}$ of the orthonormal bundle $\mathrm{O}(M, g)$

It is well known (apply, for instance [Ko], Theorem 3.2 to our case) that, taken an othonormal basis $(w_1, \ldots, w_n) \in \mathrm{O}(M, g)$, the induced orbit subset

$$S = \{(d\psi(w_1), \ldots, d\psi(w_n))/\psi \in \mathrm{Iso}(M, g)\}$$

is a closed submanifold of $\mathrm{O}(M, g)$ and it is naturally identifiable to $\mathrm{Iso}(M, g)$. As the dimension of $\mathrm{Iso}(T^2, g)$ is one, the induced vector field $\phi_*\xi$ is equal to $\pm\xi$, for all $\phi \in \mathrm{Iso}(T^2, g)$. Thus, it is clear that the orbit subset S by $\mathrm{Iso}(T^2, g)$ of an orthonormal basis with $w_1 = \xi_p/(-g(\xi_p, \xi_p))^{1/2}$ is a closed subset of $\mathcal{D}$ and, thus, compact. □

Of course, the sign of the inequality in Lemma 4.4 can be reversed in dimension 2. So, using Theorem 4.2, if $\dim(\mathrm{Iso}(T^2, g)) = 1$ then $\mathrm{Iso}(T^2, g)$ is compact and the inextendible integral curves of a (non-trivial) Killing vector field ξ are circles. Now, consider a closed curve $\gamma : \mathbb{R} \longrightarrow T^2$ of period 1, which is transversal to all these circles. Define $\Psi : \mathbb{R} \times \mathbb{R} \longrightarrow T^2$ by $\Psi(x_1, x_2) = \phi_{x_2}(\gamma(x_1)), x_1, x_2 \in \mathbb{R}$, where $\{\phi_t\}$ denotes the one-parameter group of $c\xi$, and $c > 0$ is characterized by $\phi_{t+m} = \phi_t$ for all $m \in \mathbb{Z}$. Consider the induced metric $g' = \Psi^* g$ on $\mathbb{R}^2$, whose expression in the (x_1, x_2) coordinates is clearly like a $\mathcal{G}$- or $\mathcal{G}^c$-metric. Then, one can check that Ψ is a covering map and:

Theorem 4.5. *Let (T^2, g) be a Lorentzian torus with $\dim(\mathrm{Iso}(T^2, g)) = 1$. Then (a) (T^2, g) is conformally flat if and only if one of its coverings is a $\mathcal{G}^c$-torus, and (b) g is not conformally flat if and only if one of its coverings is a $\mathcal{G}$-torus.*

As a consequence of Lemma 2.2, Theorem 2.3, Theorem 4.2 and Theorem 4.5 we have:

Corollary 4.6. *Let (T^2, g) be a Lorentzian torus admitting a (non-trivial) Killing vector field ξ. Then they are equivalent:*
(i) g is complete in one of the three causal senses,
(ii) g is complete,
(iii) g is conformally flat, and
(iv) ξ has a definite causal character.

The equivalence between (i) and (ii) in this corollary is a result on dependence of the three senses of causal completeness with no direct assumption on curvature (even though it appears indirectly in (iii)). In [La] this dependence is shown for locally symmetric semi-Riemannian manifolds. Other results on dependence can be found in [CR].

If ξ is assumed to be a (proper) conformal vector field on the Lorentzian torus (T^2, g) then the results are quite different. By Lemma 2.1, if $g(\xi, \xi)$ is a non-zero constant, then ξ is parallel and g becomes flat, but we are going to see: (a) if $g(\xi, \xi) = 0$ then g may be non-conformally flat, (b) the zeroes of ξ may be non-isolated and, if they were, their indexes may be as positive as negative, and (c) if

$g(\xi,\xi)$ has a non-definite sign, the metric g may be complete, even if ξ does not vanish at any point.

Counterexample 4.7. Consider on $\mathbb{R}^2$ the flat metric in usual coordinates $g_0 = dx_1 \otimes dx_2 + dx_2 \otimes dx_1$. Any of its conformal vector fields ξ can be written as $\xi = u(x_1)\partial/\partial x_1 + v(x_2)\partial/\partial x_2$ for some functions u, v. Thus, (c) is obtained just putting $u \equiv$ sin, $v \equiv 1$. For (b) we can take $u \equiv$ sin, $v \equiv \pm$sin to yield isolated zeros of indexes ± 1 and, multiplying them by a periodic extension of a bump function, ξ vanishes in the neighborhood of a point, without vanishing identically. For (a) change g by a $\mathcal{G}$-metric with $E = G = 0$ on $[0, 1/2]$, then choose $u \not\equiv 0$ of period 1 with support $[0, 1/2] + \mathbb{Z}$, and put $v \equiv 0$. Note also that, as every Lorentzian metric on a surface is locally conformally flat, the proof of (b) shows that its corresponding Lie algebra of conformal vector fields is always infinite dimensional.

Remark 4.8. The technique in Lemma 4.1 can also be applied to Killing vector fields on Riemannian surfaces. For them, the zeroes must be also isolated and the index at each one must be 1. Therefore, no (non-trivial) Killing vector field on a Riemannian torus can vanish at any point, and no compact Riemannian surface (M, g) of genus $l > 1$ admits Killing vector fields, that is, Iso(M, g) is finite when $l > 1$. Moreover, a non-flat Riemannian torus admitting a Killing vector field has a Riemannian covering $\mathbb{R}^2$ with a metric g_R obtained as a $\mathcal{G}^c$-metric but replacing the condition (i) of these metrics in Section 3 by $EG + F^2 < 0$, $G < 0$.

On the other hand, a conformal vector field on a Riemannian surface can have zeroes just of positive indexes (recall that the conformal diffeomorphisms close to the identity of the surface are the complex transformations of the associated 1-dimensional complex manifold). Thus, the group of conformal diffeomorphisms Conf(M, g) of a compact surface (M, g) has dimension 0 for genus $l > 1$; recall that a well-known result by Hurwitz (see, for instance, [FK]) bounds the order of Conf(M, g) by $84(l - 1)$. Even more, we have also reobtained that no conformal vector field $\xi_R(\not\equiv 0)$ of a Riemannian torus (T^2, g_R) can vanish. This can also be checked because g_R is globally conformally flat, and, thus, ξ_R is also a conformal vector field for a flat metric g'_R on T^2. A standard application of classical Bochner formula shows that ξ_R is parallel for g'_R and, thus, non-vanishing. Every conformal vector field on a Riemannian torus can be obtained by this way; therefore, its conformal group has dimension 2. On the other hand, recall that it has dimension 6 for 2-spheres.

Now we can summarize the obstructions to the corresponding Lorentzian results as follows: (a) only genus 1 must be taken into account, (b) a Lorentzian torus (even flat) may have a non-compact isometry group, (c) conformal vector fields always generate an infinite dimensional Lie algebra even on compact surfaces, and (d) Lorentzian tori may be non-conformally flat. For related results on the conformal structure of Lorentzian surfaces see [Ku2].

5. Geodesic connectedness

Consider a $\mathcal{G}_3$-metric g, and the points $(p_0, 0), (p_1, 0) \in \mathbb{R}^2$. These points can not be joined by using any geodesic $\gamma(t) = (x_1(t), x_2(t))$. In fact, if $(p_0, 0) \in \text{Im}\gamma$ we have by (3.2) that $\varepsilon D/F(p_0) \geq 0$ and if $(p_1, 0) \in \text{Im}\gamma$ then $\varepsilon D/F(p_1) \geq 0$; so, by (3.4c), $D = 0$, which implies the contradiction $r[p_1] \cap \text{Im}\gamma = \emptyset$ (case (2B) in Appendix); thus, $(\mathbb{R}^2, g)$ is not geodesically connected. Moreover, for no integers,

m_0, n_0, m_1, n_1, the points (p_0+m_0, n_0) and (p_1+m_1, n_1) can be joined by a geodesic. So, if we consider the naturally induced metric g' on the torus T^2 then $(p_0, 0)$ and $(p_1, 0)$ project onto different points, which can not be joined by any geodesic, that is:

Theorem 5.1. *For any metric $g \in \mathcal{G}_3$, the corresponding incomplete Lorentzian torus (T^2, g') is geodesically disconnected.*

Remark 5.2. Now, consider Clifton-Pohl torus (T^2, g_{CP}), which is usually constructed as a quotient of $(\mathbb{R}^2 - \{0\}, (u^2+v^2)^{-1}(du \otimes dv + dv \otimes du))$ by the properly discontinuous action by isometries generated by $(u, v) \longrightarrow (2u, 2v)$, $\forall (u, v) \in \mathbb{R}^2$, see, for example, [O, 7.16]. It has the Killing vector field $\xi = u\partial/\partial u + v\partial/\partial v$, which has a non-definite causal character (and, thus g_{CP} is incomplete). It can be seen as the $\mathcal{G}_3$-torus given by $G(x_1) = E(x_1) = -4\pi^2 \sin 4\pi x_1$, $F(x_1) = 4\pi^2 \cos 4\pi x_1$ (compare with [B, Example 3]. To check that this $\mathcal{G}_3$-torus is isometric to the Clifton-Pohl one, it suffices to take into account the covering map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2 - \{(0,0)\}$, $(x_1, x_2) \longrightarrow (u, v)$ where $u = \exp(2\pi x_2)\sin(2\pi x_1)$, $v = \exp(2\pi x_2)\cos(2\pi x_1)$.

Added in Proof. It is interesting to compare Remark 5.2 with [Sm, Section 5]. In this reference, the concept of *barrier* for a Lorentzian metric on $\mathbb{R}^2$ is studied, and the geodesic disconnectedness of Clifton-Pohl metric (in both representations above) illustrates it. □

On the other hand, for $\mathcal{G}_1$ or $\mathcal{G}_2$ metrics the behavior of the induced tori is different. By using geodesics such that $\varepsilon D/F(p_i) > 0$, $\forall i \in \mathbb{Z}$, an elementary computation shows:

Lemma 5.3. *If $g \in \mathcal{G}_1 \cup \mathcal{G}_2$, given $C_-, C_+ \in \mathbb{R}, C_- < C_+$, there exists $D \in \mathbb{R}$ such that for all $q_1, q_2 \in \mathbb{R}^2$ one can find a geodesic γ and $m, n \in \mathbb{Z}$ satisfying:*
(i) $g(\gamma', K) = D$, (ii) $g(\gamma', \gamma') \in]C_-, C_+[$ and (iii) γ joins q_1 and $q_2 + (m, n)$.

Thus, as a consequence, we have:

Theorem 5.4. *For any metric g in $\mathcal{G}_1$ or $\mathcal{G}_2$ the corresponding incomplete Lorentzian torus (T^2, g') is geodesically connected.*

6. Null geodesics

For null geodesics, recall that in any orientable and time-orientable Lorentzian surface there exist two pointwise independent null vector fields which are globally defined. The integral curves of these vector fields yield, up to reparametrizations, all the null geodesics, and they provide two foliations of the surface. For the classes $\mathcal{G}_1$ and $\mathcal{G}_2$ we can choose as independent null vector fields:

$$X_1 = G\partial/\partial x_1 + \left(F + \eta_0\sqrt{EG + F^2}\right)\partial/\partial x_2, \tag{6.1a}$$

$$X_2 = \partial/\partial x_1 + \left(\frac{F - \eta_0\sqrt{EG + F^2}}{G}\right)\partial/\partial x_2, \tag{6.1b}$$

where $\eta_0 \in \{-1, 1\}, \eta_0 F(0) > 0$ (if G vanishes at one point take the corresponding limit for X_2). Note that the former choice does work on $]p_{-1}, p_1[$ for any $\mathcal{G}'$-metric. Using (3.2) we find that the null geodesics obtained from X_1 (resp. X_2) satisfy $\varepsilon D/F(0) \leq 0$ (resp. $\varepsilon D/F(0) > 0$). Moreover, it is straightforward to check from the properties in the Appendix:

Proposition 6.1. *(1) Any null geodesic of a metric in $\mathcal{G}_3$ is incomplete.*

(2) Any null geodesic of a metric in $\mathcal{G}_1$ or $\mathcal{G}_2$ obtained from X_2 is complete.

(3) Let γ be an inextendible null geodesic in $\mathbb{R}^2$ for a metric g in $\mathcal{G}$, which has been obtained from X_1 . Set $D = g(\gamma', \xi)$,

(a) If $D = 0$ then there exists $i \in \mathbb{Z}$ such that γ is a reparametrization of $r[p_i]$. Moreover, if g lies in $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) then γ is complete (resp. incomplete).

(b) If $D \neq 0$ then γ is incomplete and there exists $i \in \mathbb{Z}$ such that γ has $r[p_i]$ as an asymptotic line. If g also lies in $\mathcal{G}_1$ or $\mathcal{G}_2$ then i can be chosen such that γ has $r[p_i]$ and $r[p_{i+1}]$ as asymptotic lines.

Remark 6.2. Using (3) in Proposition 6.1, if g is a metric in $\mathcal{G}_1$ then it has complete geodesics to which incomplete geodesics approach asymptotically. Thus, if g' is the induced metric on the torus T^2 by g, and $T(T^2)^*$ is the punctured tangent bundle of T^2, we have the next property: the subset J of $T(T^2)^*$ consisting of all the (non-zero) velocities of incomplete geodesics for g' is not closed. On the other hand, given any inextendible geodesic $\gamma(t) = (x_1(t), x_2(t))$ of $(\mathbb{R}^2, g)$ such that $x_1(0) = 0$ and $D \neq 0$, the functions x_1' and x_2' can be bounded, and then γ is complete. Nevertheless, if g lies in $\mathcal{G}_2$ then the null geodesics such that $x_1(0) = 0$ and $D = 0$ are incomplete, and J is not open. These topological properties of J were first obtained in [RS1]; in fact, Counterexample 2.2 and Counterexample 2.3 there can be seen, after taking suitable coordinates, as particular cases of $\mathcal{G}_1$ and $\mathcal{G}_2$ metrics, respectively.

7. Closed geodesics

Among all the null geodesics in $(\mathbb{R}^2, g)$ obtained from X_1 in (6.1a), only those satisfying $D = 0$ project onto closed geodesics of the induced torus (T^2, g'), Proposition 6.1(b). If g lies in $\mathcal{G}_3$ this fact also holds for all the null geodesics. To determine the remaining closed null geodesics we have:

Theorem 7.1. *Assume that g lies in $\mathcal{G}_1$ or $\mathcal{G}_2$. The following assertions for (T^2, g') are equivalent:*

(i) There is a closed null geodesic which is the projection of a null geodesic of $(\mathbb{R}^2, g)$ obtained from X_2.

(ii) Any null geodesic which is the projection of a null geodesic of $(\mathbb{R}^2, g)$ obtained from X_2 is closed.

(iii) The value of $\int_0^1 (1/G)(F - \eta_0 \sqrt{EG + F^2})$ (where $\eta_0 \in \{-1, 1\}$ is such that $F(0)\eta_0 > 0$) is a rational number.

Proof. These geodesics have $x_1' \neq 0$ at any point, thus, we can define $\bar{x}_2$ by $\bar{x}_2(x_1) = x_2(t(x_1))$. Clearly, the integral curves of X_2 project onto closed curves if and only if there exist $m, n \in \mathbb{Z}, x \in \mathbb{R}$ such that $\bar{x}(x + m) - \bar{x}(x) = n$. □

Remark 7.2. (1) Observe that, by (1B) in Appendix, $\mathcal{G}_1$-tori have a timelike or spacelike closed geodesic, and if the torus lies in $\mathcal{G}_2$ or $\mathcal{G}_3$ then both geodesics must exist. Galloway proved in [G] that a Lorentzian torus must always have a timelike or null closed geodesic. He also gave an example of Lorentzian torus without timelike closed geodesics. This example can be seen as the $\mathcal{G}'$-metric:

$$\begin{aligned} g_{gal} = & -4\pi^2 \cos^2(2\pi x + \pi/4) dx^2 \\ & - 2\pi^2 \sin(2\pi x + \pi/4)(dx \otimes dy + dy \otimes dx) + \cos^2(2\pi x + \pi/4) dy^2. \end{aligned}$$

It is easy to check that a geodesic for a $\mathcal{G}'$-metric can project onto a closed geodesic of T^2 just in the Appendix cases (1), (2A), and the last of (2C)(d). So, we can generalize the metric g_{gal} defining a new family $\mathcal{G}_{gal}$ of $\mathcal{G}'$-metrics, by: (i) $G \leq 0$ (timelike geodesics can not lie in the cases (1) and (2C)(d)), and (ii) $F(p_0)F(p_1) < 0$ (no geodesic lie in the case (2A)).

(2) Consider now a $\mathcal{G}^c$-metric. Clearly, the vector fields X_1, X_2 are well defined on $\mathbb{R}^2$, and Theorem 7.1 also works for these metrics. Moreover, the geodesics obtained from X_1 are now complete and a result analogous to Theorem 7.1 (changing the $-$ sign in the integral to a $+$) also works for them. In particular, we obtain that there exists a closed null geodesic if and only if its corresponding foliation is by circles. Of course this last assertion was expected, because it clearly happen in the flat case, which have the same null pregeodesics that the conformally flat ones.

8. Complete geodesics with diverging velocity

A straightforward sufficient condition to obtain completeness for inextendible geodesics on a torus, is to check that the components of its velocity are bounded (see, for instance, [RS1]). However, this property is not necessary for complete geodesics.

Theorem 8.1. *Let g be a metric in $\mathcal{G}'$ and $\gamma(t) = (x_1(t), x_2(t))$ be an inextendible non-null geodesic in $(\mathbb{R}^2, g)$ such that $D = 0$. Then x_2' is not bounded and γ is complete (resp. incomplete) whenever g lies in $\mathcal{G}_1$ (resp. g lies in $\mathcal{G}_2$ or $\mathcal{G}_3$).*

Proof. The assertion on x_2' is obvious from equations (3.2), and note that there is $i \in \mathbb{Z}$ such that $p_i < x_1(t) < p_{i+1}$ for all t. Thus, fixed $x_0 \in]p_i, p_{i+1}[$ we get

$$t(x_1) - t(x_0) = \varepsilon \int_{x_0}^{x_1} \sqrt{\frac{EG + F^2}{GC}(x)}dx, \quad \text{for all } x_1 \in]p_i, p_{i+1}[.$$

The result follows from the following fact: when x_1 approaches p_i or p_{i+1} then the value of the last integral is ∞ if G' vanishes at p_i or p_{i+1}, respectively; otherwise, its value is finite (see the case (2B) in Appendix). ☐

Remark 8.2. (1) Theorem 8.1 yields complete geodesics (on a compact manifold) which have a diverging velocity, in the sense that the velocity is not contained in any compact subset of the tangent bundle. Clearly, this property can not hold for a Riemannian connection.

(2) Recall that these complete geodesics are found in $\mathcal{G}_1$-tori, which are incomplete. However, it is possible to construct complete and compact Lorentzian manifolds with geodesics showing the mentioned property in dimensions greater than two [RS3, Remark 3.18].

9. D'Alembert's equation

We will consider now (inhomogeneous) D'Alembert's equation for a Lorentzian torus (T^2, g), that is, the equation in u:

$$\Box_g u = f, \tag{9.1}$$

where $\Box_g$ ($\equiv$ divergence of the gradient) denotes the D'Alembertian for g, and f is a continuous function on $T^2 (\equiv \mathbb{R}^2/\mathbb{Z}^2)$ with $\int_{T^2} f \mu_g = 0$, μ_g being the volume element for g.

Assume first that f depends just on the first variable, and $f \not\equiv 0$. If we consider the flat metrics $g_0 = dx_1 \otimes dx_2 + dx_2 \otimes dx_1$ and $g'_0 = dx_1^2 - dx_2^2$, with D'Alembertians $\Box_{g_0} = \partial^2/\partial x_1 \partial x_2$, $\Box_{g'_0} = \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2$, then clearly the corresponding equation (9.1) does not have any solution for g_0, but it does for g'_0. The reason for this difference can be seen as follows: in both cases f is invariant under the one-parameter group of one of the two (parallel) vector fields which generate T^2 from $\mathbb{R}^2$; and this vector field is null in the first case, but non-null in the second one. This fact leads to

Corollary 9.1. *Let (T^2, g) be a complete Lorentzian torus with* $\dim(\mathrm{Iso}(T^2, g)) = 1$ *and f any function on T^2 with total integral zero. If f is invariant under the one-parameter group of isometries of g then D'Alembert's (inhomogeneous) equation* (9.1) *admits a solution.*

Proof. Consider a Killing vector field $\xi \not\equiv 0$, and the flat metric $g^* = |g(\xi, \xi))|^{-1} g$. Equation (9.1) for g is equivalent for solving

$$\Box_{g^*} u = |g(\xi, \xi)| f.$$

Now, observe that the last member is invariant under the one-parameter group of the non-null g^*-parallel vector field ξ, which is a generator of T^2. □

Moreover, one can check that the explicit form of $\mathcal{G}^c$-tori in Section 3 allows us to obtain these solutions to (9.1) directly.

Remark 9.2. (1) The analogous result for incomplete tori does not hold. Recall that, if k and k^* are the Gaussian curvatures of two pointwise conformal Lorentzian metrics g and g^*, where $g^* = \exp(2\omega) g$ for some $\omega \in C^2(T^2)$, then $\Box_g \omega = k - \exp(2\omega) k^*$ on T^2. Thus, if g is a $\mathcal{G}$-metric then k is invariant under its one-parameter isometry group, but the equation $\Box_g u = k$ has no solution.

(2) One could hope that, given f, the solvability of (9.1) could depend on the curvature of g. But this is not clear, in fact, consider the $\mathcal{G}$- and $\mathcal{G}^c$-metrics with $E \equiv 0$. If we choose the coefficient G equal to a function G_0 to yield a $\mathcal{G}$-metric, g_1, then adding a suitable constant $\lambda > |G_0|$ we can obtain a conformally flat metric g_2, with $G = G_0 + \lambda$. As the Gaussian curvature of these two metrics is $(1/F)(G'/2F)'$ (that is, it depends just on F and the derivatives of G) the metrics g_1 and g_2 have the same curvature k, even though the corresponding equation (9.1) with $f = k$ has solution just in the first case.

Now, we can state a result on prescribed curvature on T^2 in conditions analogous to [Bu, Section 2]. First, it is straightforward to check,

Lemma 9.3. *Let k be a continuous* 1*-periodic function on $\mathbb{R}$ which changes sign in* $[0, 1]$*. Then there exists a (smooth) positive* 1*-periodic function F on $\mathbb{R}$ such that* $\int_0^1 kF = 0$.

Then, our (constructive) result is (compare with Theorem 2.3 in [Bu])

Theorem 9.4. *Let k be a* 1*-periodic function on $\mathbb{R}$ which changes sign in* $[0, 1]$*. Then there exist two Lorentzian metrics $g_1 \in \mathcal{G}, g_2 \in \mathcal{G}^c$ such that their Gaussian curvature at each point (x_1, x_2) is $k(x_1)$.*

Proof. Just take g_1 and g_2 putting $E \equiv 0$, F as in Lemma 9.3, and the derivative of G as $G'(x_1) = G'(0) F(x_1)/F(0) + 2F(x_1) \int_0^{x_1} kF$, for all $x_1 \in \mathbb{R}$, where $G'(0)$ is chosen to get $\int_0^1 G' = 0$. Then, set two suitable values of $G(0)$ and compute the corresponding curvature as for g_1, g_2 in Remark 9.2(2). □

On the other hand, we point out that the problem of finding a Riemannian or Lorentzian metric on T^2 with prescribed curvature can be restated with forms as follows. Let $\mathrm{Mod}_R(T^2)$ be the quotient of the Riemannian metrics on T^2 obtained by identifying those which are globally pointwise conformal. Then, for each 2-form Ω on T^2 with $\int_{T^2}\Omega = 0$ and each $\mathcal{C} \in \mathrm{Mod}_R(T^2)$ there exists a $g_R \in \mathcal{C}$ such that its canonically associated volume μ_R satisfies

$$k_R \mu_R = \Omega, \tag{9.2}$$

where k_R is the Gaussian curvature of g_R. The result can be extended to any compact connected orientable 2-dimensional manifold M, if we assume $\int_M \Omega = 2\pi\mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Euler characteristic of M (see [WW]).

Now, if we consider the quotient $\mathrm{Mod}_L(T^2)$ of pointwise globally conformal Lorentzian metrics on T^2, the analogous equation

$$k_L \mu_L = \Omega, \tag{9.3}$$

has no solution when $\Omega = 0$ in the class $\mathcal{C}$ of any null incomplete metric g_L. So, it appears as a natural open question to characterize the classes of Lorentzian metrics on T^2 and exact 2-forms Ω for which equation (9.3) can be solved.

Appendix: General Behaviour of the geodesics in $\mathcal{G}'$-tori

Let $\gamma : I \longrightarrow \mathbb{R}^2$, $I =]a_-, a_+[$, $-\infty \le a_- < a_+ \le +\infty$, an inextendible non-constant geodesic of $(\mathbb{R}^2, g)$, $g \in \mathcal{G}'$, $\gamma(t) = (x_1(t), x_2(t))$, for all $t \in I$, and put $C = g(\gamma', \gamma')$, $D = g(\gamma', \partial/\partial x_2)$, as before; set also for each $x \in \mathbb{R}$, $r[x] = \{(x,s)/s \in \mathbb{R}\}$. Then, it is not difficult to check from equations (3.2) and (3.3) that:

(1) The first component of γ is constant if and only if one of the next two conditions (1A), (1B) holds:

(1A) $C = 0$, $D = 0$. In this case, $\exists i \in \mathbb{Z}$ such that γ is a reparametrization of $r[p_i]$ and:

(a) If $G'(p_i) = 0$, then γ is complete.

(b) If $G'(p_i) \neq 0$, then γ is incomplete. Moreover, given $t_1 \in I$, put $\eta \in \{-1, 1\}$ equal to the sign of $(x_2'(t_1)G'(p_i)F(p_i))$, then $a_\eta = \eta\infty$, $|a_{-\eta}| < \infty$.

(1B) $C \neq 0$, $D \neq 0$, $\exists p \in \mathbb{R}$, such that $G(p) \neq 0$, $G'(p) = 0$ and γ is an affine reparametrization of $r[p]$. In particular, γ is complete.

(2) If the first component of γ is not constant, let $t_0 \in I$ be such that $x_1'(t_0) \neq 0$, $x_0 = x_1(t_0)$, put $\varepsilon \in \{-1, 1\}$ such that $\varepsilon x_1'(t_0) > 0$, and define

$$x_+ = \mathrm{Sup}\{x \in \mathbb{R}/x_0 < x, D^2 + GC|_{[x,x_0]} > 0\},$$
$$x_- = \mathrm{Inf}\{x \in \mathbb{R}/x < x_0, D^2 + GC|_{[x,x_0]} > 0\},$$

which belong to $\mathbb{R} \cup \{\pm\infty\}$. If there exists $i \in \mathbb{Z}$ such that $\varepsilon D/F(p_i) \le 0$, put also

$$p_+ = \mathrm{Min}\{p_i/i \in \mathbb{Z}, x_0 < p_i, \varepsilon D/F(p_i) \le 0\},$$
$$p_- = \mathrm{Max}\{p_i/i \in \mathbb{Z}, p_i < x_0, \varepsilon D/F(p_i) \le 0\}.$$

Then γ lies in exactly one of the next three cases:

(2A) $\varepsilon D/F(p_i) > 0$, $\forall i \in \mathbb{Z}$, and, then, γ is complete.

(2B) $D = 0$. It is equivalent to $x_+ = p_+$ and also to $x_- = p_-$. In this case:

(i) When $t \to a_{+\varepsilon}$ (resp. $t \to a_{-\varepsilon}$) then γ tends asymptotically to $r[p_+]$ (resp. $r[p_-]$).

(ii) γ is defined until $-\varepsilon\infty$ ($a_{-\varepsilon} = -\varepsilon\infty$) if and only if $G'(x_-) = 0$, and until $\varepsilon\infty$ if and only if $G'(x_+) = 0$; so, γ is complete if and only if $G'(x_+) = G'(x_-) = 0$ (see also Theorem 8.1).

(2C) $\exists i \in \mathbb{Z} : \varepsilon D/F(p_i) < 0$. Then, we have $\mathrm{Im}\gamma \cap r[p_i] = \emptyset$, and γ lies in one of these cases:

(a) Incomplete towards both extremes: $x_- < p_- < x_0 < p_+ < x_+$. In this case, $-\infty < a_- < a_+ < +\infty$, and when $t \to a_\varepsilon$ (resp. $t \to a_{-\varepsilon}$) then γ tends asymptotically towards $r[p_+]$ (resp. $r[p_-]$).

(b) Incomplete towards p_-: $x_- < p_- < x_0 < x_+ < p_+$. In this case, $|a_{-\varepsilon}| < \infty$, and when $t \to a_{-\varepsilon}$ we have that γ tends asymptotically towards $r[p_-]$. Moreover:

If $G'(x_+) = 0$ then $a_\varepsilon = \varepsilon\infty$, and when $t \to \varepsilon\infty$, γ tends asymptotically to $r[x_+]$.

If $G'(x_+) \neq 0$ then $a_\varepsilon \neq \varepsilon\infty$, and when $t \to a_\varepsilon$, γ tends asymptotically to $r[p_-]$.

(c) Incomplete towards p_+: $p_- < x_- < x_0 < p_+ < x_+$. Analogous to the former case.

(d) Complete: $p_- < x_- < x_0 < x_+ < p_+$. That is, $]a_-, a_+[=] - \infty, +\infty[$ and, moreover,

If $G'(x_+) = 0$, $G'(x_-) = 0$, then when $t \to \varepsilon\infty$, (resp. $t \to -\varepsilon\infty$), γ tends asymptotically to $r[x_+]$ (resp. $r[x_-]$).

If $G'(x_+) \neq 0$, $G'(x_-) = 0$, then when $t \to \varepsilon\infty$ or $t \to -\varepsilon\infty$, γ tends asymptotically to $r[x_-]$.

If $G'(x_+) = 0$, $G'(x_-) \neq 0$, then when $t \to \varepsilon\infty$ or $t \to -\varepsilon\infty$), γ tends asymptotically to $r[x_+]$.

If $G'(x_+) \neq 0$, $G'(x_-) \neq 0$, then when $t \to \varepsilon\infty$ or $t \to -\varepsilon\infty$, the image of γ oscillates between $r[x_+]$ and $r[x_-]$. □

Note that incomplete geodesics appear in the cases (1A) (b), (2C) (a),(b),(c) and sometimes in (2B). In the cases (2C) (a),(b),(c) and (2B) we can define:

$$\nu_+ = \lim_{x \to p_+, x < p_+} \frac{F(x)/|F(x)|}{G(x)/|G(x)|}, \quad \text{(resp.} \quad \nu_- = \lim_{x \to p_-, x > p_-} \frac{F(x)/|F(x)|}{G(x)/|G(x)|})$$

which determine if the geodesic with asymptote $r[p_+]$ (resp. $r[p_-]$), tends to it in the positive or negative x_2-direction. In the case (1A)(b) we can define

$$\nu = -\frac{F(p_i)/|F(p_i)|}{G'(p_i)/|G'(p_i)|}$$

which satisfies: when t tends to the finite extreme of the domain of x_2 (that is, where γ is incomplete) then $x_2(t)$ tends to $\nu\infty$.

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